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SEACAS Theory Manuals: Part II.
Nonlinear Continuum Mechanics

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Abstract

This report summarizes the key continuum mechanics concepts required for the systematic prescription and numerical solution of finite deformation solid mechanics problems. Topics surveyed include measures of deformation appropriate for media undergoing large deformations, stress measures appropriate for such problems, balance laws and their role in nonlinear continuum mechanics, the role of frame indifference in description of large deformation response, and the extension of these theories to encompass two dimensional idealizations, structural idealizations, and rigid body behavior. There are three companion reports that describe the problem formulation, constitutive modeling, and finite element technology for nonlinear continuum mechanics systems.

Nonlinear Continuum Mechanics: Table of Contents

Introduction

Overview

In this report we examine in detail the continuum mechanical issues necessary for rigorous specification of large deformation problems in solid mechanics. The discussion will provide a bridge between the generic problem statement given at the close of Formulation of Nonlinear Problems and the in-depth presentation of constitutive theory to be discussed in Constitutive Modeling. At the close of the latter report, we will be in a position to turn attention to numerical methods as applied to large deformation solid mechanics.

The current report's presentation is organized as follows. We begin with a discussion of large deformation kinematics, including consideration of velocity and acceleration measures and the quantification of deformation and deformation rates in a general context. We then discuss the various measures of stress that are frequently encountered in large deformation analysis. With these preliminaries in hand, we will then be in a position to state the relevant balance laws in notation appropriate for large deformation problems. We will also at this point discuss the important concept of material frame indifference, which demands that material laws be unaltered by rigid body motions. We will see that this concept places important restrictions on the kinematic and stress measures that are suitable for prescription of constitutive laws, providing important background information for a subsequent report.

The above information will be presented in a three-dimensional notational framework, assuming that the solids of interest are likewise fully three-dimensional continua. Formulations appropriate for two-dimensional problems and for structural entities in three dimensions can be readily deduced from these equations. Accordingly, we will briefly present the modifications necessary to adapt our theory to two-dimensional geometries and to problems possessing axial symmetry. Also we will discuss how continuum mechanical descriptions of structural elements, including shells and beams, can be deduced from the three-dimensional formalism. We will also briefly examine how rigid bodies can be incorporated into the notational structure we propose.

It should be emphasized that although many of the concepts to be discussed in this chapter are applicable to Eulerian formulations, the presentation is targeted primarily toward Lagrangian description of boundary value problems. Furthermore, for notational simplicity we work almost exclusively in Cartesian coordinate systems rather than in general curvilinear coordinates (some deviation from this is obviously necessary when axisymmetry is discussed). The interested reader may care to consult [Fung, Y.C., 1965] for discussion of such curvilinear formulations in a small-strain context, and [Marsden, J.E. and Hughes, T.J.R., 1983] for their rigorous extension to large deformation problems.

Measures of Deformation

We continue using the notation from the last report (Formulation of Nonlinear Problems) that was presented schematically in Figure 1.7. We restrict our attention to some time $t \in (0, T)$, and consider the corresponding configuration mapping φ_t, which can be mathematically represented via $\varphi_t : \bar{\Omega} \to \Re^3$. The *deformation gradient* $\mathbf{F}$ is given by the gradient of this transformation, i.e.:

$$\mathbf{F} = \frac{\partial \varphi_t}{\partial \mathbf{X}}, \tag{2.1}$$

or in indices:

$$F_{iJ} = \frac{\partial \varphi_{ti}}{\partial X_J}. \tag{2.2}$$

In (2.2) one may notice a notational feature we will use unless otherwise noted: lower case indices are to be associated with coordinates in the spatial frame, while upper case indices are associated with material coordinates. Repeated indices in expressions will continue to imply summation.

The deformation gradient is the most basic object used to quantify the local deformation at a point in a solid. Most kinematic measures and concepts we will discuss rely on it explicitly or implicitly for their definitions. For example, we can use our knowledge of elementary calculus to give an interpretation of the determinant of $\mathbf{F}$. Consider a cube of material in the reference configuration (see Figure 2.1) whose sides can be assumed to be aligned with the coordinate axes X_I, $I = 1, 2, 3$. The initial differential volume dV of this cube is given by

$$dV = dX_1 dX_2 dX_3. \tag{2.3}$$

If we now consider the condition of this cube of material after the deformation φ_t is applied, we notice that its volume in the current configuration dv is that of the parallelepiped spanned by the three vectors $\varphi_t(\overrightarrow{dX_J})$, where the notation $\overrightarrow{dX_J}$ is used to indicate a reference vector in coordinate direction J with magnitude dX_J. This volume can be written in terms of the vector triple product:

$$dv = \varphi_t(\overrightarrow{dX_1}) \cdot (\varphi_t(\overrightarrow{dX_2}) \times \varphi_t(\overrightarrow{dX_3})). \tag{2.4}$$

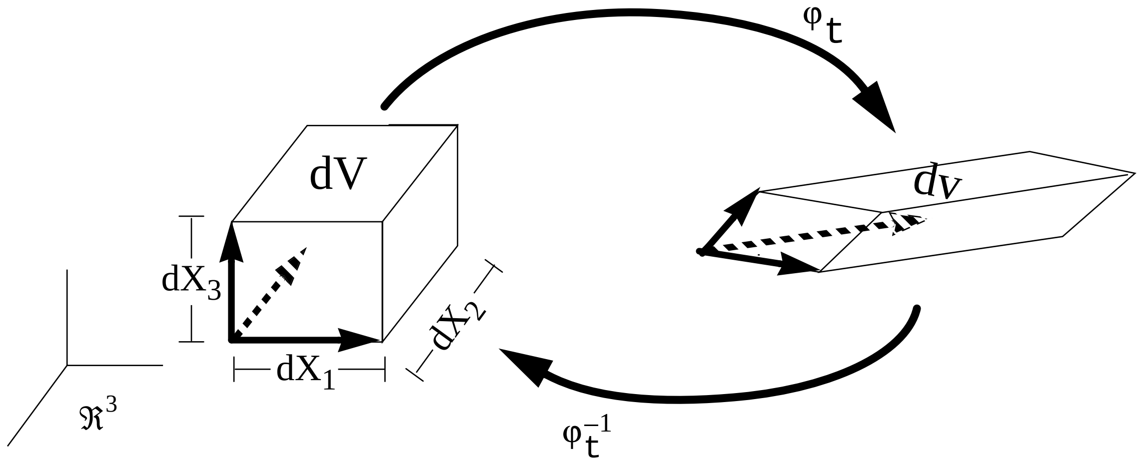

Figure 2.1 Concept of volume change: deformation of a volume element as described by the configuration mapping φ_t.

If we consider any differential vector $\overrightarrow{dR}$ in the reference configuration, the calculus of differentials tells us that application of the mapping φ_t will produce a differential vector $\overrightarrow{dr} = \varphi_t(\overrightarrow{dR})$ whose coordinates are given via

$$(\overrightarrow{dr})_i = \frac{\partial \varphi_{ti}}{\partial X_K}(\overrightarrow{dR})_K. \tag{2.5}$$

Application of this logic to the particular differential vectors $\overrightarrow{dX_J}$ leads one to conclude:

$$(\varphi_t(\overrightarrow{dX_J}))_i = \begin{cases} F_{i1}dX_1, J = 1 \\ F_{i2}dX_2, J = 2 \\ F_{i3}dX_3, J = 3 \end{cases}. \tag{2.6}$$

We can write (2.4) in indicial notation by first noting that the cross product of two vectors **a** and **b** is written as

$$(\mathbf{a} \times \mathbf{b})_i = e_{ijk}a_j b_k, \tag{2.7}$$

where e_{ijk}, the *permutation symbol,* has the following interpretation:

$$e_{ijk} = \begin{cases} 1 \text{ if } (i, j, k) = (1,2,3) \text{ or } (2,3,1) \text{ or } (3,1,2) \\ -1 \text{ if } (i, j, k) = (3,2,1) \text{ or } (2,1,3) \text{ or } (1,3,2) \\ 0 \text{ otherwise} \end{cases}. \tag{2.8}$$

Equation (2.4) is then reexpressed via

$$\begin{aligned} dv &= F_{i1}dX_1(e_{ijk}F_{j2}dX_2F_{k3}dX_3) \\ &= e_{ijk}F_{i1}F_{j2}F_{k3}dX_1dX_2dX_3 = \det(\mathbf{F})dV, \end{aligned} \tag{2.9}$$

where we have used Eq. (2.3) and the fact that $\det(\mathbf{F}) = e_{ijk}F_{i1}F_{j2}F_{k3}$ (which can be verified through actual trial). Introducing the notation $J = \det(\mathbf{F})$, we conclude

$$dv = JdV. \tag{2.10}$$

Equation (2.10) tells us that the deformation φ_t converts reference differential volumes dV to current volumes dv according to the determinant of the deformation gradient. For this mapping to make physical sense, the current volume dv should be positive which then places a physical restriction upon $\mathbf{F}$ that must be obeyed pointwise throughout the medium:

$$J = \det(\mathbf{F}) = \det\left(\frac{\partial\varphi}{\partial\mathbf{X}}\right) > 0. \tag{2.11}$$

This physical restriction has important mathematical consequences as well. According to the inverse function theorem of multivariate calculus, a smooth function whose gradient has a nonzero determinant possesses a smooth and differentiable inverse. Since we have assumed φ_t to be smooth and physical restrictions demand that $J \neq 0$, we can conclude that a function φ_t^{-1} exists that is differentiable; in fact, the gradient of this function is given by

$$\frac{\partial\varphi_t^{-1}}{\partial\mathbf{x}} = \mathbf{F}^{-1}. \tag{2.12}$$

We will assume throughout the remainder of our discussion that $J > 0$, so that such an inverse is guaranteed to exist.

With the definition of $\mathbf{F}$ in hand, we turn our attention to the quantification of local deformation in a body. For any matrix, such as $\mathbf{F}$, whose determinant is positive, the following decompositions can always be made:

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}. \tag{2.13}$$

In (2.13) $\mathbf{R}$ is a proper orthogonal tensor (right-handed rotation), while $\mathbf{U}$ and $\mathbf{V}$ are positive definite and symmetric tensors. One can show that under the conditions stated the decompositions in (2.13) can always be made and that, in fact, they are unique. The interested reader should consult [Gurtin, M.E., 1981], Chapter 1 for details. The decompositions in (2.13) are called right and left polar decompositions of $\mathbf{F}$, respectively. $\mathbf{R}$ is often called the rotation tensor, while $\mathbf{U}$ and $\mathbf{V}$ are sometimes referred to as the right and left stretches.

The significance of the polar decomposition is made more clear in Figure 2.2, where we consider the deformation of a neighborhood of material surrounding a point $\mathbf{X} \in \Omega$. Equation (2.5) shows us that the full deformation gradient maps arbitrary reference differentials into their current positions at time t; this idea also applies to neighborhoods of material having infinitesimal extent. By considering the polar decomposition, we see that this deformation of material neighborhoods can always be conceived as consisting of two parts. Considering the right polar decomposition as an example, $\mathbf{U}$ contains all information necessary to describe the distortion of a neighborhood of material, while $\mathbf{R}$ then maps this distorted volume into the current configuration through pure (right-handed) rotation. In consideration of the left decomposition, the rotation $\mathbf{R}$ is considered first, followed by the distortion $\mathbf{V}$. In developing measures of local deformation, we can then concentrate our attention on either $\mathbf{U}$ or $\mathbf{V}$. The choice of which decomposition to use is typically based on the coordinates in which we wish to write strains: the right stretch $\mathbf{U}$ most naturally takes reference coordinates as arguments, while the left stretch $\mathbf{V}$ is ordinarily written in terms of spatial coordinates. We might indicate this explicitly via

$$\mathbf{F}(\mathbf{X}) = \mathbf{R}(\mathbf{X})\mathbf{U}(\mathbf{X}) = \mathbf{V}(\varphi(\mathbf{X}))\mathbf{R}(\mathbf{X}). \tag{2.14}$$

In characterizing large deformations, it is convenient also to define the right and left Cauchy-Green tensors via

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} \tag{2.15}$$

and

$$\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}. \tag{2.16}$$

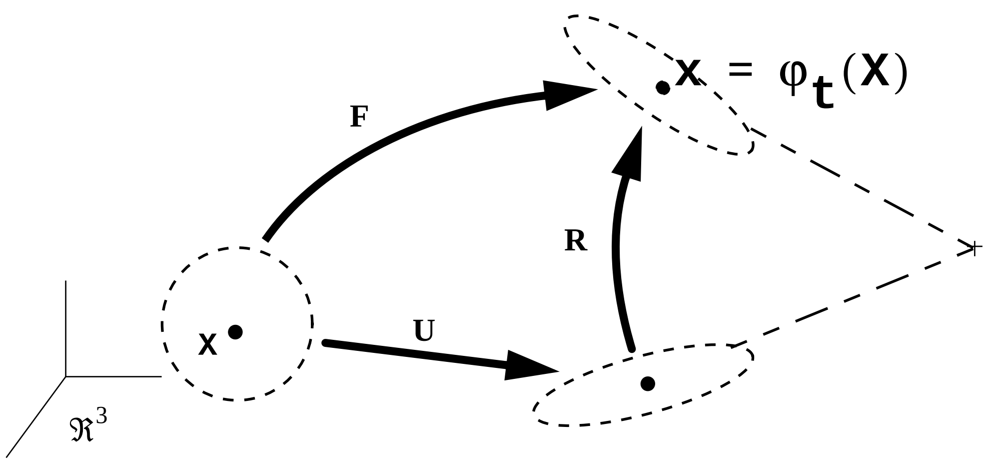

Figure 2.2 Physical interpretation of the polar decomposition. (Dotted outline indicates a neighborhood of point X.)

The right Cauchy-Green tensor is ordinarily considered to be a material object (i.e., $\mathbf{C}(\mathbf{X})$), while the left Cauchy-Green tensor is a spatial object ($\mathbf{B}(\varphi_t(\mathbf{X}))$. Since $\mathbf{R}$ is orthogonal, one can write

$$\mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}, \qquad (2.17)$$

where $\mathbf{I}$ is the 3x3 identity tensor. Using this fact and manipulating Eqs. (2.14)-(2.16) also reveals that

$$\mathbf{U} = \mathbf{C}^{\frac{1}{2}} \qquad (2.18)$$

and

$$\mathbf{V} = \mathbf{B}^{\frac{1}{2}}. \qquad (2.19)$$

One can see the point of connection with the small strain theory by considering the *Green strain tensor* $\mathbf{E}$, defined with respect to the reference configuration:

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}). \qquad (2.20)$$

Let us define a reference configuration displacement field $\mathbf{u}$, such that

$$\mathbf{u}(\mathbf{X}) = \varphi(\mathbf{X}) - \mathbf{X}. \qquad (2.21)$$

Working in indicial notation, let us attempt to write $\mathbf{E}$ in terms of $\mathbf{u}$:

$$\begin{aligned} E_{IJ} &= \frac{1}{2}(C_{IJ} - \delta_{IJ}) = \frac{1}{2}(F_{iI}F_{iJ} - \delta_{IJ}) \\ &= \frac{1}{2}\left(\frac{\partial}{\partial X_I}(u_i + X_i)\frac{\partial}{\partial X_J}(u_i + X_i) - \delta_{IJ}\right) \\ &= \frac{1}{2}\left(\left(\frac{\partial u_i}{\partial X_I} + \delta_{iI}\right) + \left(\frac{\partial u_i}{\partial X_J} + \delta_{iJ}\right) - \delta_{IJ}\right) \\ &= \frac{1}{2}\left(\delta_{iI}\frac{\partial}{\partial X_J}(u_i) + \delta_{iJ}\frac{\partial}{\partial X_I}(u_i) + \frac{\partial u_i \partial u_i}{\partial X_I \partial X_J}\right) \end{aligned} . \qquad (2.22)$$

In the case where the displacement gradients are small, i.e., $\left|\frac{\partial u_i}{\partial X_J}\right| \ll 1$, the quadratic term in (2.22) will be much smaller than the terms linear in the displacement gradients. If, in addition, the displacement components u_i are very small when compared with the size of the body, then the distinction between reference and spatial coordinates becomes unnecessary and Eq. (2.22) simplifies to

$$E_{IJ} \approx \frac{1}{2}\left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I}\right), \tag{2.23}$$

which is recognized as being identical with the infinitesimal case (c.f. Eq. (1.56)).

Rates of Deformation

Introduction

The development of the last section fixed our attention on an instant $t \in (0, T)$, and proposed some measurements of material deformation in terms of the configuration mapping φ_t. We now allow time to vary and consider two questions: 1) how velocities and accelerations are quantified in both the spatial and reference frames; and 2) how time derivatives of deformation measures are properly considered in a large deformation framework. The former topic is obviously crucial in the formulation of dynamics problems, while the latter is necessary, for example, in rate-dependent materials where quantities, such as strain rate, must be quantified.

Material and Spatial Velocity and Acceleration

One obtains the *material velocity* $\mathbf{V}$ and the *material acceleration* $\mathbf{A}$ by fixing attention on a particular material particle (i.e., fixing the reference coordinate $\mathbf{X}$), and then considering successive (partial) time derivatives of the motion $\varphi(\mathbf{X}, t)$. This can be written mathematically as

$$\mathbf{V}(\mathbf{X}, t) = \frac{\partial}{\partial t}\varphi(\mathbf{X}, t) \tag{2.24}$$

and

$$\mathbf{A}(\mathbf{X}, t) = \frac{\partial}{\partial t}\mathbf{V}(\mathbf{X}, t) = \frac{\partial^2}{\partial t^2}(\varphi(\mathbf{X}, t)). \tag{2.25}$$

Note in Eqs. (2.24) and (2.25) that $\mathbf{V}$ and $\mathbf{A}$ take $\mathbf{X}$ as their first argument; hence their designation as material quantities. A Lagrangian description of motion, in which reference coordinates are the independent variables, would most naturally use these measures of velocity and acceleration.

An Eulerian description, on the other hand, would, in general, require measures written in terms of points $\mathbf{x}$, without requiring explicit knowledge of material points $\mathbf{X}$. The *spatial velocity* $\mathbf{v}$ and the *spatial acceleration* $\mathbf{a}$ are obtained from (2.24) and (2.25) through a change of variables:

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{V}(\varphi_t^{-1}(\mathbf{x}), t) = \mathbf{V}_t \bullet \varphi_t^{-1}(\mathbf{x}) \tag{2.26}$$

and

$$\mathbf{a}(\mathbf{x}, t) = \mathbf{A}(\varphi_t^{-1}(\mathbf{x}), t) = \mathbf{A}_t \bullet \varphi_t^{-1}(\mathbf{x}). \tag{2.27}$$

The expression given in (2.27) for the spatial acceleration may be unfamiliar to those readers versed in fluid mechanics who may be more accustomed to thinking of acceleration as the *total time derivative* of the spatial velocity $\mathbf{v}$. We reconcile these different viewpoints here through the introduction of the equivalent concept of the *material time derivative*, defined, in general, as the time derivative of any object, spatial or material, taken so that the identity of the material particle is held fixed. Applying this concept to the spatial velocity gives:

$$\begin{aligned} \mathbf{a}(\mathbf{x}, t) &= \dot{\mathbf{v}}(\mathbf{x}, t)\big|_{\mathbf{x} = \varphi(\mathbf{X}, t)} \\ &= \frac{d}{dt}\bigg|_{\mathbf{X}\text{ fixed}} \mathrm{v}(\varphi((\mathrm{X}, t), t)) \\ &= \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}(\mathbf{x}, t) \cdot \frac{\partial \varphi}{\partial t}(\varphi_t^{-1}(\mathbf{x}), t) + \frac{\partial \mathbf{v}}{\partial t}(\varphi_t^{-1}(\mathbf{x}), t)\right). \\ &= \left(\frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{v} \cdot \mathbf{v}\right) \end{aligned} \tag{2.28}$$

This may be recognized as the so-called “total time derivative” of the spatial velocity $\mathbf{v}$. Exercising the concept of a material time derivative a little further, we can see from (2.24) that the material velocity is the material time derivative of the motion, so that

$$\mathbf{V} = \dot{\varphi}. \tag{2.29}$$

Comparing Eqs. (2.25) and (2.28), we can also conclude that $\mathbf{A}$ and $\mathbf{a}$ are, in fact, the same physical entity expressed in different coordinates. The former is most naturally written in terms of $\mathbf{V}$, while the latter is conveniently expressed in terms of $\mathbf{v}$.

One may see in (2.28) the superposed dot notation for the time derivative of $\mathbf{v}$. Such superposed dots will always imply a material time derivative in this text, whether applied to material quantities or, as in this case, spatial ones. It is further emphasized that the gradient $\nabla \mathbf{v}$ is taken with respect to spatial coordinates and is, therefore, called the *spatial velocity gradient*. It is used often enough to warrant a special symbol which we denote as $\mathbf{L}$:

$$\mathbf{L} = \nabla \mathbf{v}. \tag{2.30}$$

Rate of Deformation Tensors

From the spatial gradient $\mathbf{L}$ defined in (2.30), we can define two spatial tensors $\mathbf{D}$ and $\mathbf{W}$, known respectively as the *spatial rate of deformation tensor* and the *spatial spin tensor*:

$$\mathbf{D} = \nabla_s \mathbf{v} = \frac{1}{2}[\mathbf{L} + \mathbf{L}^T], \tag{2.31}$$

and

$$\mathbf{W} = \nabla_a \mathbf{v} = \frac{1}{2}[\mathbf{L} - \mathbf{L}^T]. \tag{2.32}$$

It is clear that $\mathbf{D}$ is merely the symmetric part of the velocity gradient, while $\mathbf{W}$ is the antisymmetric, or skew, portion.

The quantities $\mathbf{D}$ and $\mathbf{W}$ are spatial measures of deformation. $\mathbf{D}$ is effectively a measure of strain rate suitable for large deformations, while $\mathbf{W}$ provides a local measure of the rate of rotation of the material. In fact, it is readily verified that in small deformations, Eq. (2.31) amounts to nothing more that the time derivative of the infinitesimal strain tensor defined in (1.56). It is of interest at this point to discuss whether appropriate material counterparts of these objects exist. Toward this end let us calculate the material time derivative of the *deformation gradient* $\mathbf{F}$, noting in so doing that if $\mathbf{F}$ is an analytic function, then the order of partial differentiation can be reversed:

$$\dot{\mathbf{F}} = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial \mathbf{X}}\varphi(\mathbf{X}, t)\right] = \frac{\partial}{\partial \mathbf{X}}\left[\frac{\partial}{\partial t}\varphi(\mathbf{X}, t)\right] = \frac{\partial \mathbf{V}}{\partial \mathbf{X}}. \tag{2.33}$$

From (2.33) we conclude that the material time derivative $\dot{\mathbf{F}}$ is nothing more than the material velocity gradient. Manipulating this quantity further we find

$$\begin{aligned}\frac{\partial \mathbf{V}}{\partial \mathbf{X}} &= \frac{\partial}{\partial \mathbf{X}}(\mathbf{v}\circ\varphi_t) = \nabla\mathbf{v}(\varphi_t(\mathbf{X}))\frac{\partial}{\partial \mathbf{X}}(\varphi_t(\mathbf{X})) \\ &= \mathbf{L}(\varphi_t(\mathbf{X}))\mathbf{F}(\mathbf{X})\end{aligned}. \tag{2.34}$$

Examination of (2.33) and (2.34) reveals that

$$\mathbf{L} = (\dot{\mathbf{F}}\circ\varphi_t^{-1})\mathbf{F}^{-1}. \tag{2.35}$$

Recalling the definition for the right Cauchy-Green strain tensor $\mathbf{C}$ in Eq. (2.15), we compute its material time derivative via:

$$\begin{aligned}\dot{\mathbf{C}} &= \frac{\partial}{\partial t}[\mathbf{F}^T\mathbf{F}] = \dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}} \\ &= (\mathbf{L}\mathbf{F})^T\mathbf{F} + \mathbf{F}^T(\mathbf{L}\mathbf{F}) = \mathbf{F}^T(\mathbf{L} + \mathbf{L}^T)\mathbf{F}\end{aligned}, \tag{2.36}$$

which in view of (2.31), leads us to conclude

$$\dot{\mathbf{C}}(\mathbf{X}, t) = 2\mathbf{F}^T(\mathbf{X}, t)\mathbf{D}(\varphi_t(\mathbf{X}), t)\mathbf{F}(\mathbf{X}, t). \tag{2.37}$$

In view of (2.37) $\frac{1}{2}\dot{\mathbf{C}}$ is sometimes called the *material rate of deformation tensor.*

Noting that $\mathbf{F}$ is the Jacobian of the transformation φ_t, readers with a background in differential geometry will recognize $\frac{1}{2}\dot{\mathbf{C}}$ as the pull-back of the spatial tensor field $\mathbf{D}$ defined on $\varphi_t(\Omega)$. Conversely, $\mathbf{D}$ is the push-forward of the material tensor field $\frac{1}{2}\dot{\mathbf{C}}$ defined on Ω. The concepts of pull-back and push-forward are outside the scope of our present investigation, but the basic physical principle they embody in the current context is perhaps useful. Loosely speaking, the push-forward (or pull-back) of a tensor with respect to a given transformation produces a tensor in the new frame of reference that we, as observers, would observe as identical to the original tensor if we were embedded in the material during the transformation. Thus the same physical principle is represented by both $\frac{1}{2}\dot{\mathbf{C}}$ and $\mathbf{D}$, but they are very different objects mathematically since the transformation that interrelates them is the deformation itself. Recalling the definition of Green's strain $\mathbf{E}$ given in Eq. (2.20), we can easily see that

$$\dot{\mathbf{E}} = \frac{1}{2}\dot{\mathbf{C}} = \mathbf{F}^{\mathrm{T}}\mathbf{D}\mathbf{F}. \tag{2.38}$$

This further substantiates the interpretation of $\mathbf{D}$ as a strain rate as suggested earlier.

We have thus far developed measures of strain and strain rate appropriate for both the spatial and reference configurations. Although it is not clear at this point why other measures may be needed, let us consider appropriate definitions of these quantities for the rotated configuration defined according to the polar decomposition and depicted schematically in Figure 2.2. This can be readily done by extending the idea of pull-back and push-forward as discussed above, by applying to the linear transformation $\mathbf{R}$ relating the rotated configuration to the spatial one.

The *rotated rate of deformation tensor* $\boldsymbol{D}$ is, therefore, defined via:

$$\begin{aligned}\boldsymbol{D}(\mathbf{X}, t) &= \mathbf{R}^{\mathrm{T}}(\mathbf{X}, t) \cdot \mathbf{D}(\varphi(\mathbf{X}, t), t) \cdot \mathbf{R}(\mathbf{X}, t) \\ &= \mathbf{R}^{\mathrm{T}}(\mathbf{D}\circ\varphi)\mathbf{R}\end{aligned}. \tag{2.39}$$

Noting that

$$\dot{\mathbf{C}} = 2\mathbf{F}^{\mathrm{T}}(\mathbf{D}\circ\varphi)\mathbf{F} = 2\mathbf{U}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}(\mathbf{D}\circ\varphi)\mathbf{R}\mathbf{U} = 2\mathbf{U}^{\mathrm{T}}\boldsymbol{D}\,\mathbf{U}, \tag{2.40}$$

we find

$$\boldsymbol{D} = \frac{1}{2}\mathbf{U}^{-1}\dot{\mathbf{C}}\mathbf{U}^{-1} = \frac{1}{2}\mathbf{C}^{-1/2}\dot{\mathbf{C}}\mathbf{C}^{-1/2}. \tag{2.41}$$

In connection with the rotated reference frame, another tensor, $\boldsymbol{L}$, is sometimes introduced:

$$\boldsymbol{L} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}. \tag{2.42}$$

As shown below, note that $\boldsymbol{L}$ is skew:

$$\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} + \mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}} = \frac{\partial}{\partial t}(\mathbf{R}\mathbf{R}^{\mathrm{T}}) = \frac{\partial \mathbf{I}}{\partial t} = 0\,. \tag{2.43}$$

We will return later in this report to the various measures associated with the rotated configuration. They have particular importance in the study of material frame indifference.

Stress Measures

In this section we discuss the quantification of force intensity, or stress, within a body undergoing potentially large amounts of deformation. We begin with the Cauchy stress tensor $\mathbf{T}$, and note that provided we associate this object with the spatial configuration, this object can be interpreted exactly as in the infinitesimal case outlined in Linear Elastic IBVP. In the current notational framework, we interpret the components of $\mathbf{T}$, which we shall denote as T_{ij}, as representing forces per unit areas in the spatial configuration at a given spatial point $\mathbf{x} \in \varphi_{\mathbf{t}}(\Omega)$.

It will be necessary in our study to consider related measures of stress defined in terms of the other configurations we have discussed, particularly the reference and rotated configurations. To motivate this discussion, let us reconsider the concept of traction discussed previously in the context of the infinitesimal elastic system. The reader may recall that given a plane passing through the point of interest $\mathbf{x}$, the traction, or force per unit area acting on this plane, is given by the formula

$$t_i = T_{ij} n_j , \tag{2.44}$$

where n_j is the unit normal vector to the plane in question.

Let us consider two differential vectors, $d\mathbf{r}_1$ and $d\mathbf{r}_2$, in such a plane passing through the spatial point $\mathbf{x}$, as indicated in Figure 2.3. We assume that $d\mathbf{r}_1$ and $d\mathbf{r}_2$ are linearly independent from one another and that both differential vectors have $\mathbf{x}$ as their base point. We further assume that their orientations are such that the following relation from basic geometry holds:

$$d\mathbf{r}_1 \times d\mathbf{r}_2 = \mathbf{n}\, da , \tag{2.45}$$

where da is the (differential) area of the parallelogram encompassed by $d\mathbf{r}_1$ and $d\mathbf{r}_2$.

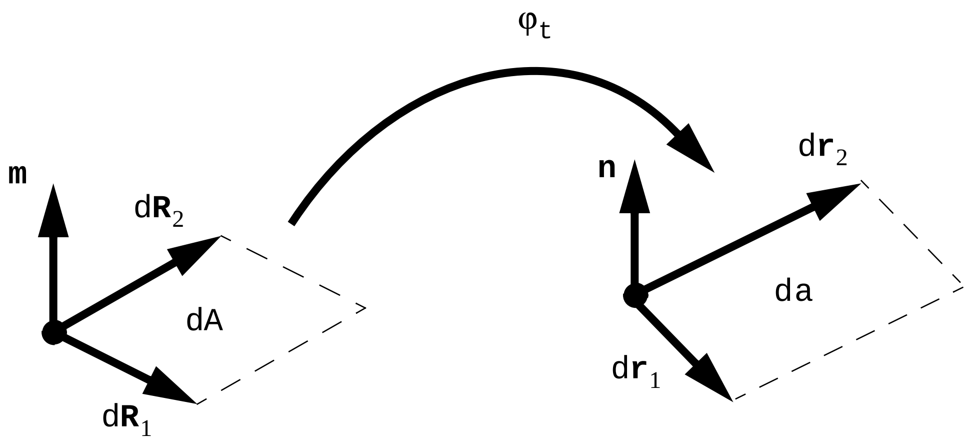

Figure 2.3 Notation for derivation of Nanson's formula.

As in the discussion in Measures of Deformation (see Eq. (2.5)), we can think of the differential vectors $\mathrm{d}\mathbf{r}_1$ and $\mathrm{d}\mathbf{r}_2$ as the current positions of reference differential vectors $\mathrm{d}\mathbf{R}_1$ and $\mathrm{d}\mathbf{R}_2$, which are based at $\mathbf{X} = \varphi_{\mathbf{t}}^{-1}(\mathbf{x})$. In indicial notation we can relate these two sets of differential vectors using the deformation gradient via:

$$(\mathrm{d}\mathbf{r}_1)_i = F_{iI}(\mathrm{d}\mathbf{R}_1)_I, \tag{2.46}$$

and

$$(\mathrm{d}\mathbf{r}_2)_i = F_{iI}(\mathrm{d}\mathbf{R}_2)_I. \tag{2.47}$$

We now seek to reexpress (2.45) in terms of reference quantities. Working in indicial notation we can write

$$\begin{aligned} n_i \mathrm{d}a &= e_{ijk}F_{jJ}(\mathrm{d}\mathbf{R}_1)_J F_{kK}(\mathrm{d}\mathbf{R}_2)_K \\ &= e_{ljk}\delta_{li}F_{jJ}(\mathrm{d}\mathbf{R}_1)_J F_{kK}(\mathrm{d}\mathbf{R}_2)_K \\ &= e_{ljk}F_{lL}F^{-1}_{Li}F_{jJ}(\mathrm{d}\mathbf{R}_1)_J F_{kK}(\mathrm{d}\mathbf{R}_2)_K \end{aligned} \quad . \tag{2.48}$$

Let us extract and work with a particular product in the last line of Eq. (2.48), namely $e_{ljk}F_{lL}F_{jJ}F_{kK}$. One can show by a case-by-case examination that the following relation holds:

$$e_{ljk}F_{lL}F_{jJ}F_{kK} = e_{LJK}e_{ljk}F_{l1}F_{j2}F_{k3}. \tag{2.49}$$

The reader may recall from Measures of Deformation that $J = \det(\mathbf{F})$ has the following representation in indicial notation:

$$J = \det(\mathbf{F}) = e_{ljk}F_{l1}F_{j2}F_{k3}. \tag{2.50}$$

Combination of Eqs. (2.48), (2.49), and (2.50) yields the following result:

$$\begin{aligned} n_i da &= J e_{LJK} F^{-1}_{Li} (d\mathbf{R}_1)_J (d\mathbf{R}_2)_K \\ &= J F^{-1}_{Li} m_L dA \end{aligned} . \tag{2.51}$$

In Eq. (2.51) dA is the differential reference area spanned by $d\mathbf{R}_1$ and $d\mathbf{R}_2$, and $\mathbf{m}$ is the reference unit normal to this area.

In direct notation we can express this result as

$$\mathbf{n} da = J\mathbf{F}^{-T}\mathbf{m} dA. \tag{2.52}$$

Equation (2.52) is sometimes referred to as Nanson's formula and it is important, among other reasons, because it provides the appropriate change-of-variables formula for surface integrals in the reference and current configurations. In the current context we are more interested in computing the product of the traction acting on our plane at $\mathbf{x}$ and the differential area under consideration. Denoting this differential force by $d\mathbf{f}$, we may write

$$d\mathbf{f} = \mathbf{t} da = \mathbf{T}\mathbf{n} da = J\mathbf{T}\mathbf{F}^{-T}\mathbf{m} dA. \tag{2.53}$$

In examining (2.53) we find that the following definition is useful

$$\mathbf{P}(\mathbf{X}) = J(\mathbf{X})\mathbf{T}(\varphi_t(\mathbf{X}))\mathbf{F}^{-T}(\varphi_t(\mathbf{X})), \tag{2.54}$$

which then allows us to write

$$d\mathbf{f} = \mathbf{P}\mathbf{m} dA. \tag{2.55}$$

In examining Eq. (2.55), we note that the product $\mathbf{Pm}$ represents a traction, with the physical interpretation of current force divided by reference area. The stress $\mathbf{P}$ is called the (*First*) *Piola-Kirchhoff Stress*, and like the associated Piola traction, $\mathbf{Pm}$, measures stress by referencing the force acting on areas to the magnitude of those areas in their undeformed configurations. The one-dimensional manifestation of this stress measure is the engineering stress, σ_E, originally defined in Eq. (1.3).

in the sense discussed in Rates of Deformation, it is worthy to note that $\mathbf{P}$ is neither a pure spatial nor a reference object. Such an object can be constructed by performing a pull-back of the spatial Cauchy stress tensor $\mathbf{T}$ to the reference configuration:

$$\begin{aligned} \mathbf{S}(\mathbf{X}) &= J\mathbf{F}^{-1}(\varphi_t(\mathbf{X}))\mathbf{T}(\varphi_t(\mathbf{X}))\mathbf{F}^{-T}(\varphi_t(\mathbf{X})) \\ &= \mathbf{F}^{-1}(\varphi_t(\mathbf{X}))\mathbf{P}(\mathbf{X}) \end{aligned} . \tag{2.56}$$

$\mathbf{S}$ is called the *Second Piola-Kirchhoff* stress tensor and it is a purely reference object. We note in particular that $\mathbf{S}$ is a symmetric tensor, while $\mathbf{P}$ is not symmetric, in general.

This same concept of pull-back can be employed to define a stress tensor in the rotated configuration, which we shall denote as $\boldsymbol{T}$. This rotated stress tensor is defined via:

$$\boldsymbol{T}(\varphi_t(\mathbf{X})) = \mathbf{R}^T(\varphi_t(\mathbf{X}))\mathbf{T}(\varphi_t(\mathbf{X}))\mathbf{R}(\varphi_t(\mathbf{X})). \qquad (2.57)$$

As was the case with the rotated configuration quantities introduced in Rate of Deformation Tensors, this definition will be of particular importance in the subsequent examination of frame indifference.

Balance Laws

Introduction

In this section we examine the local forms of the various conservations laws as expressed in the various reference frames (spatial, reference, and rotated) we have introduced. To expedite our development, we first discuss how integral representations of balances can be converted to pointwise conservation principles, a process known as localization.

Localization

Suppose we consider an arbitrary volume of material, $V \subset \Omega$, in the reference configuration of a solid body, as depicted in Figure 2.4. Suppose further that we can establish the following generic integral relation over this volume:

$$\int_V f(\mathbf{X})dV = 0, \tag{2.58}$$

where f is some reference function, be it scalar, vector, or tensor-valued, defined over all of Ω. Suppose now that (2.58) holds true for each and every subvolume V of Ω. The localization theorem then states that

$$f = 0 \text{ pointwise in } \Omega. \tag{2.59}$$

The interested reader should consult [Gurtin, M.E., 1981], Section 5 for elaboration on this principle. It should be noted that the same procedure can be applied spatially. In other words, if we are working with a spatial object, we might consider arbitrary volumes v in the spatial domain, and if the following holds for a spatial object g for all v:

$$\int_v g(\mathbf{x})dv = 0, \tag{2.60}$$

then $g(\mathbf{x}) = 0$ throughout $\varphi_t(\Omega)$.

Our primary interest in these localization principles will be to take the well-known conservation laws for control volumes and convert them to their local counterparts valid pointwise throughout the medium.

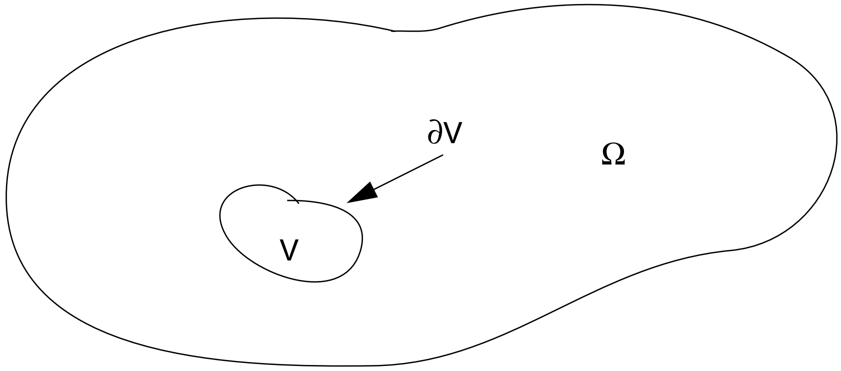

Figure 2.4 Notation for localization concept.

Conservation of Mass

In consideration of the conservation of mass, let us consider a fixed control volume, v, in the spatial domain, completely filled with our solid body at the instant in question as the body moves through it. We may write a conservation of mass for this control volume via

$$-\int_{\partial v} \rho \mathbf{v} \cdot \mathbf{n} da = \int_{v} \frac{\partial \rho}{\partial t} dv, \tag{2.61}$$

where the term on the left can be interpreted as the net mass influx to the control volume, and the right-hand side is the rate of mass accumulation inside the control volume. Applying the divergence theorem to the left-hand side gives

$$-\int_{v} \nabla \cdot (\rho \mathbf{v}) dv = \int_{v} \frac{\partial \rho}{\partial t} dv. \tag{2.62}$$

This can be further rearranged to yield

$$\int_{v} \left(\frac{\partial \rho}{\partial t} + \nabla \rho \cdot \mathbf{v} + \rho (\nabla \cdot \mathbf{v}) \right) dv = 0, \tag{2.63}$$

which can be established for any arbitrary spatial volume v. Applying the localization theorem gives the local expression of continuity, which may be familiar to those versed in fluid mechanics:

$$\frac{\partial \rho}{\partial t} + \nabla \rho \cdot \mathbf{v} + \rho (\nabla \cdot \mathbf{v}) = \dot{\rho} + \rho (\nabla \cdot \mathbf{v}) = 0, \tag{2.64}$$

where the concept of the material time derivative has been employed (cf. Eq. (2.28)).

A reference configuration representation of continuity is also highly desirable, especially in the study of solid mechanics. Therefore, we convert (2.63) to a reference configuration integral and obtain:

$$\int_{V = \varphi_t^{-1}(v)} (\dot{\rho} + \rho \dot{\mathbf{F}} : \mathbf{F}^{-T}) J \, dV = 0 , \tag{2.65}$$

where the transformation between dv and dV is accomplished using (2.10); and the chain rule is used to convert $\nabla \cdot \mathbf{v}$ via

$$\begin{aligned} v_{i,i}(\mathbf{x}) &= \frac{\partial}{\partial x_i} V_i(\varphi_t^{-1}(\mathbf{x})) \\ &= \frac{\partial}{\partial X_I} V_i(\varphi_t^{-1}(\mathbf{x})) \frac{\partial X_I}{\partial x_i}(\varphi_t^{-1}(\mathbf{x})) , \\ &= \dot{F}_{iI}(\varphi_t^{-1}(\mathbf{x})) F_{Ii}^{-1}(\varphi_t^{-1}(\mathbf{x})) \end{aligned} \tag{2.66}$$

which the reader will recognize as the indicial notation form of $\dot{\mathbf{F}} : \mathbf{F}^{-T}$. Applying the localization theorem in the reference configuration gives

$$\dot{\rho} J + \rho J \dot{\mathbf{F}} : \mathbf{F}^{-T} = 0 , \tag{2.67}$$

which holds pointwise in Ω.

Working in indicial notation we can work further to simplify (2.67) by concentrating on the term $J \dot{\mathbf{F}} : \mathbf{F}^{-T}$. Let us compute the material time derivative of J as follows:

$$\dot{J} = \frac{\partial J}{\partial F_{mM}} \dot{F}_{mM} . \tag{2.68}$$

Calculation of $\frac{\partial J}{\partial F_{mM}}$ is achieved via

$$
\begin{aligned}
\frac{\partial J}{\partial F_{mM}} &= \frac{\partial}{\partial F_{mM}}(e_{ijk}F_{i1}F_{j2}F_{k3}) \\
&= e_{ijk}\delta_{im}\delta_{M1}F_{j2}F_{k3} \\
&\quad + e_{ijk}\delta_{jm}\delta_{M2}F_{i1}F_{k3} + e_{ijk}\delta_{km}\delta_{M3}F_{i1}F_{j2}, \\
&= e_{ijk}F_{iN}F^{-1}_{Nm}\delta_{M1}F_{j2}F_{k3} \\
&\quad + e_{ijk}F_{jN}F^{-1}_{Nm}\delta_{M2}F_{i1}F_{k3} \\
&\quad + e_{ijk}F_{kN}F^{-1}_{Nm}\delta_{M3}F_{i1}F_{j2}
\end{aligned}
\tag{2.69}
$$

which can be further simplified to yield

$$
\begin{aligned}
\frac{\partial J}{\partial F_{mM}} &= JF^{-1}_{1m}\delta_{M1} + JF^{-1}_{2m}\delta_{M2} + JF^{-1}_{3m}\delta_{M3} \\
&= JF^{-1}_{Im}\delta_{MI} = JF^{-1}_{Mm}
\end{aligned}
. \tag{2.70}
$$

Substitution into (2.68) gives

$$\dot{J} = JF^{-1}_{Mm}\dot{F}_{mM}, \tag{2.71}$$

which is nothing more than the indicial form of

$$\dot{J} = J\mathbf{F}^{-T}:\dot{\mathbf{F}}. \tag{2.72}$$

Substitution into (2.67) gives

$$\dot{\rho}J + \rho\dot{J} = \frac{d}{dt}(\rho J) = 0. \tag{2.73}$$

Equation (2.73) is the reference configuration version of the continuity equation and tells us that the product of the density and deformation gradient determinant must be invariant with time for all material points. This is commonly enforced in practice by assigning a reference density ρ_0 to all material points. If the current density ρ is always computed via

$$\rho = \frac{1}{J}\rho_0, \tag{2.74}$$

then Eq. (2.73) is automatically satisfied (recall that the Jacobian J is unity in the reference configuration).

Conservation of Linear Momentum

Considering once more a fixed control volume v, the control volume balance of linear momentum can be expressed as

$$\int_{\partial v} (\rho \mathbf{v})\mathbf{v} \cdot \mathbf{n} da + \int_{v} \frac{\partial}{\partial t}(\rho \mathbf{v}) dv = \int_{v} \mathbf{f} dv + \int_{\partial v} \mathbf{t} da . \tag{2.75}$$

The first term on the left expresses the momentum outflux, while the second represents the rate of accumulation inside the control volume. This net change of momentum is produced by the total resultant force on the system, equal to the sum effect of the body force $\mathbf{F}$ and the surface tractions $\mathbf{t}$.

Applying the divergence theorem to both surface integrals, we find

$$\int_{\partial v} (\rho \mathbf{v})\mathbf{v} \cdot \mathbf{n} da = \int_{v} [\nabla \cdot (\rho \mathbf{v})\mathbf{v} + \rho(\nabla \mathbf{v})\mathbf{v}] dv , \tag{2.76}$$

and

$$\int_{\partial v} \mathbf{t} da = \int_{\partial v} \mathbf{T}\mathbf{n} da = \int_{v} \nabla \cdot \mathbf{T} dv . \tag{2.77}$$

Substituting (2.76) and (2.77) into (2.75) and rearranging gives

$$\int_{v} \begin{bmatrix} \nabla \cdot \mathbf{T} + \mathbf{f} - \rho \dfrac{\partial \mathbf{v}}{\partial t} - \rho(\nabla \mathbf{v})\mathbf{v} \\ -\dfrac{\partial \rho}{\partial t}\mathbf{v} - (\nabla \rho \cdot \mathbf{v})\mathbf{v} - \rho(\nabla \cdot \mathbf{v})\mathbf{v} \end{bmatrix} dv = 0 . \tag{2.78}$$

Employing the spatial form of the continuity equation (Eq.(2.63)) and recalling the formula for the material time derivative (Eq. (2.28)) gives

$$\int_{v} [\nabla \cdot \mathbf{T} + \mathbf{f} - \rho \dot{\mathbf{v}}] dv = 0 . \tag{2.79}$$

By the localization theorem this implies

$$\nabla \cdot \mathbf{T} + \mathbf{f} = \rho \dot{\mathbf{v}} \tag{2.80}$$

pointwise, which is recognized as the same statement of linear momentum balance utilized in our earlier treatment of linear elasticity.

In large deformation problems it is desirable to also have a reference configuration form of (2.80). Converting (2.79) to its indicial form we have

$$\int_V [T_{ij,j} + f_i - \rho \dot{v}_i] dv = 0 . \tag{2.81}$$

Working with the stress divergence term first we write

$$T_{ij,j} = \frac{\partial T_{ij}}{\partial X_J}\frac{\partial X_J}{\partial x_j} = \frac{\partial T_{ij}}{\partial X_J} F^{-1}_{Jj} . \tag{2.82}$$

Using Eq. (2.54) we can write

$$\begin{aligned} \frac{\partial T_{ij}}{\partial X_J} &= \frac{\partial}{\partial X_J}\left(\frac{1}{J} P_{iI} F_{jI}\right) \\ &= \frac{-1}{J^2}\frac{\partial J}{\partial F_{kK}}\frac{\partial F_{kK}}{\partial X_J} P_{iI} F_{jI} + \frac{1}{J}\frac{\partial}{\partial X_J}(P_{iI} F_{jI}) \end{aligned} . \tag{2.83}$$

Using Eq. (2.70) we can simplify (2.83) and postmultiply by F^{-1}_{Jj} to obtain:

$$\frac{\partial T_{ij}}{\partial X_J} F^{-1}_{Jj} = \frac{-1}{J} F^{-1}_{Kk}\frac{\partial F_{kK}}{\partial X_J} P_{iJ} + \frac{1}{J}\frac{\partial P_{iI}}{\partial X_I} + \frac{1}{J} F^{-1}_{Jj}\frac{\partial F_{jI}}{\partial X_J} P_{iI} . \tag{2.84}$$

The first and last terms on the right-hand side of (2.84) cancel each other due to the fact that $\frac{\partial F_{jI}}{\partial X_J} = \frac{\partial F_{jJ}}{\partial X_I}$. Therefore, we have

$$\frac{\partial T_{ij}}{\partial X_J} F^{-1}_{Jj} = \frac{1}{J}\frac{\partial P_{iI}}{\partial X_I} . \tag{2.85}$$

Using this result and applying a change of variables to (2.81) gives

$$\int_V (P_{iI,I} + F_i - \rho_0 \dot{V}_i) dV = 0 , \tag{2.86}$$

where $F_i = J f_i$, the prescribed body force per unit reference volume. Employing the localization theorem gives

$$\mathrm{DIV}\,\mathbf{P} + \mathbf{F} = \rho_0 \dot{\mathbf{V}} \tag{2.87}$$

pointwise in Ω, which expresses the balance of linear momentum in terms of reference coordinates. In (2.87) we have used the notation DIV to indicate the divergence operator applied in reference coordinates.

Conservation of Angular Momentum

Considering once more an arbitrary control volume in the spatial frame, we can write its balance of angular momentum via

$$\int_{\partial v}(\mathbf{x}\times\rho\mathbf{v})\mathbf{v}\cdot\mathbf{n}da+\int_{v}\frac{\partial}{\partial t}(\mathbf{x}\times\rho\mathbf{v})dv = \int_{v}(\mathbf{x}\times\mathbf{f})dv+\int_{\partial v}\mathbf{x}\times\mathbf{t}da, \tag{2.88}$$

where the terms on the left-hand side are the outflux and accumulation terms, while the terms on the right-hand side represent the total resultant torque.

Working this time in indicial notation, we apply the divergence theory to the surface integrals as follows:

$$\int_{\partial v}e_{ijk}\rho x_j v_k v_l n_l da = \int_{v}\begin{pmatrix}\rho_{,l}e_{ijk}x_j v_k v_l + e_{ijk}\rho\delta_{jl}v_k v_l \\ e_{ijk}\rho x_j v_{k,l}v_l + e_{ijk}\rho x_j v_k v_{l,l}\end{pmatrix}dv, \tag{2.89}$$

and

$$\int_{\partial v}e_{ijk}x_j T_{kl}n_l da = \int_{v}(e_{ijk}x_j T_{kl,l}+e_{ijk}T_{kj})dv. \tag{2.90}$$

Substituting (2.89) and (2.90) into (2.88) and rearranging terms reveals that:

$$\int_{v}\begin{pmatrix} e_{ijk}x_j\left(T_{kl,l}+f_k-\rho\dfrac{\partial v_k}{\partial t}-\rho\dfrac{\partial v_k}{\partial x_l}v_l\right) \\ -e_{ijk}x_j v_k\left(\dfrac{\partial\rho}{\partial t}+\dfrac{\partial\rho}{\partial x_l}v_l+\rho v_{l,l}\right) \\ +e_{ijk}T_{kj}-\rho e_{ijk}v_j v_k\end{pmatrix}dv = 0. \tag{2.91}$$

Using Eqs. (2.81) and (2.64) and noting that the cross product of a vector with itself is zero, we can simplify Eq. (2.91) and apply the localization theorem to conclude

$$e_{ijk}T_{kj} = 0, \tag{2.92}$$

which, in turn, implies the following three equations:

$$T_{23} = T_{32},\ T_{13} = T_{31},\ T_{21} = T_{12}. \tag{2.93}$$

In other words, the symmetry of the Cauchy stress tensor is a direct consequence of the conservation of angular momentum. Use of Eqs. (2.56) and (2.57), respectively, easily reveals that the Second Piola Kirchhoff stress $\mathbf{S}$ and the rotated stress tensor $\boldsymbol{T}$ are likewise symmetric.

The First Piola Kirchhoff stress is not symmetric and is not, in fact, a tensor in the purest sense since it does not fully live in either the spatial or reference frame.

Stress Power

Finally, we examine the consequences of a control volume expression of energy balance. We assume herein a purely mechanical description and assume, to begin, that there is no mechanical dissipation, so that the system we consider conserves energy exactly. In other words, all work put into the system through the applied loads goes either into stored internal elastic energy or into kinetic energy.

With this in mind the conservation of energy for a spatial control volume is written as

$$\begin{aligned}&\int_{\partial v}\left(e+\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}\right)\mathbf{v}\cdot\mathbf{n}da+\int_{v}\frac{\partial}{\partial t}\left(e+\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}\right)dv\\&=\left(\int_{v}\mathbf{f}\cdot\mathbf{v}dv+\int_{\partial v}(\mathbf{Tn})\cdot\mathbf{v}da\right)\end{aligned}, \quad (2.94)$$

where e is the internal stored energy (i.e., elastic energy) per unit spatial volume.

As we have done previously, we apply the divergence theorem to the surface integrals:

$$\begin{aligned}&\int_{\partial v}\left(e+\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}\right)\mathbf{v}\cdot\mathbf{n}da\\&=\int_{v}\begin{bmatrix}\nabla\cdot\mathbf{v}\left(e+\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}\right)+\nabla e\cdot\mathbf{v}\\+\frac{1}{2}\nabla\rho\cdot\mathbf{v}(\mathbf{v}\cdot\mathbf{v})+\rho\mathbf{v}\cdot(\nabla\mathbf{v})\mathbf{v}\end{bmatrix}dv\end{aligned}, \quad (2.95)$$

and

$$\int_{\partial v}\mathbf{t}\cdot\mathbf{v}da=\int_{v}[\mathbf{T}:\nabla\mathbf{v}+(\nabla\cdot\mathbf{T})\cdot\mathbf{v}]dv. \quad (2.96)$$

Substituting (2.95) and (2.96) into (2.94) and rearranging gives

$$0=\int_{v}\begin{bmatrix}\left(\nabla\cdot\mathbf{T}+\mathbf{f}-\rho\frac{\partial\mathbf{v}}{\partial t}-\rho(\nabla\mathbf{v})\mathbf{v}\right)\cdot\mathbf{v}\\-\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\left(\frac{\partial\rho}{\partial t}+\rho(\nabla\cdot\mathbf{v})+\nabla\rho\cdot\mathbf{v}\right)\\+\mathbf{T}:\nabla\mathbf{v}-e(\nabla\cdot\mathbf{v})-\dot{e}\end{bmatrix}dv. \quad (2.97)$$

Using Eqs. (2.81) and (2.64) we find

$$0 = \int_v [\mathbf{T}:\nabla\mathbf{v} - e(\nabla \cdot \mathbf{v}) - \dot{e}]dv. \tag{2.98}$$

Splitting (2.98) into two integrals we have

$$0 = \int_v \mathbf{T}:\nabla\mathbf{v}dv - \int_v (e(\nabla \cdot \mathbf{v}) - \dot{e})dv. \tag{2.99}$$

We now wish to convert (2.99) to the reference configuration and apply localization. In so doing we recognize that the second integral in (2.99) can be treated directly analogously to that of Eq. (2.63), with the density in (2.63) being replaced by the energy e in the current case. The result of this manipulation will lead to a term form identical to the result (2.73), with e substituted for ρ. In other words, we have

$$\int_v (e(\nabla \cdot \mathbf{v}) - \dot{e})dv = \int_V \frac{d}{dt}(eJ)dV. \tag{2.100}$$

Concentrating on the first integral and using Eqs. (2.35) and (2.68) to aid in the calculation, we find

$$\begin{aligned} \int_v \mathbf{T}:\nabla\mathbf{v}dv &= \int_V (\mathbf{T}\circ\varphi^{-1}):(\mathbf{L}\circ\varphi^{-1})JdV \\ &= \int_V (\mathbf{T}\circ\varphi^{-1}):(\dot{\mathbf{F}}\mathbf{F}^{-1})JdV = \int_V \mathbf{P}:\dot{\mathbf{F}}dV \end{aligned}. \tag{2.101}$$

Combining these results and employing the localization theorem, we conclude that

$$\frac{d}{dt}(eJ) = \dot{E} = \mathbf{P}:\dot{\mathbf{F}} \tag{2.102}$$

pointwise in Ω, where E is the stored elastic energy per unit reference volume. Therefore, $\mathbf{P}:\dot{\mathbf{F}}$ represents the rate of energy input into the material by the stress (per unit volume), commonly known as the *stress power*. Taking into account the various measures of stress and deformation rate we have considered, it can be shown that for a given material point, the stress power can be written in the following alternative forms:

$$\text{Stress power} = \mathbf{P}:\dot{\mathbf{F}} = \left(\frac{1}{2}\mathbf{S}\dot{\mathbf{C}}\right) = J\mathbf{T}:\mathbf{D} = J\boldsymbol{T}:\boldsymbol{D}. \tag{2.103}$$

It should be noted that this definition can be used also for dissipative (i.e., nonconservative) materials but that the interpretation becomes different: the stress power in this case is the sum of the rate of increase of stored energy and the rate of energy dissipation by the solid.

Frame Indifference

Frame Indifference

An important concept to be considered in the formulation of constitutive theories in large deformations is that of frame indifference, alternatively referred to as objectivity. Although somewhat mathematically involved, the concept of objectivity is fairly simple to understand physically.

When we write constitutive laws in their most general forms, we seek to express tensoral quantities, such as stress and stress rate, in terms of kinematic tensoral quantities, most commonly strain and strain rate. The basic physical idea behind frame indifference is that this constitutive relationship should be unaffected by any rigid body motions the material may be undergoing at the instant in question. Mathematically we describe this situation by defining an alternative reference frame that is rotating and translating with respect to the coordinate system in which we pose the problem. For our constitutive description to make sense, the tensoral quantities we use in it (stress, stress rate, strain, and strain rate) should simply transform according to the laws of tensor calculus when subjected to this transformation. If a given quantity does this we say it is material frame indifferent, and if it does not we say it is not properly invariant.

Consider now a motion, $\varphi(\mathbf{X}, t)$. We imagine ourselves to be viewing this motion from another reference frame, denoted in the following by *, which can be related to the original spatial frame via

$$\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x}, \tag{2.104}$$

where $\mathbf{x} = \varphi(\mathbf{X}, t)$. In (2.104) $\mathbf{c}(t)$ is a relative rigid body translation between the original frame and observer *, while a relative rotation is produced by the proper orthogonal tensor $\mathbf{Q}(t)$. To observer * the motion appears as defined by

$$\mathbf{x}^* = \varphi^*(\mathbf{X}, t) = \mathbf{c}(t) + \mathbf{Q}(t)\varphi(\mathbf{X}, t). \tag{2.105}$$

Then for the * frame, we can define an appropriate deformation gradient:

$$\mathbf{F}^* = \frac{\partial}{\partial \mathbf{X}}\varphi_t^* = \mathbf{Q}\frac{\partial}{\partial \mathbf{X}}\varphi_t(\mathbf{X}) = \mathbf{QF} \tag{2.106}$$

and a spatial velocity gradient $\mathbf{L}^*$:

$$\begin{aligned}\mathbf{L}^* &= \nabla^*\mathbf{v}^* = \dot{\mathbf{F}}^*(\mathbf{F}^*)^{-1} = \frac{d}{dt}(\mathbf{QF})(\mathbf{QF})^{-1} \\ &= (\dot{\mathbf{Q}}\mathbf{F}\mathbf{F}^{-1}\mathbf{Q}^{\mathsf{T}} + \mathbf{Q}\nabla\mathbf{v}\mathbf{F}\mathbf{F}^{-1}\mathbf{Q}^{\mathsf{T}})\end{aligned}, \tag{2.107}$$

which can be simplified to

$$\mathbf{L}^* = \mathbf{QLQ}^T + \dot{\mathbf{Q}}\mathbf{Q}^T. \tag{2.108}$$

For $\mathbf{L} = \nabla\mathbf{v}$ to be objective, it would transform according to the laws of tensor transformation between the two frames, so that only the first term on the right-hand side of (2.108) would be present. Clearly $\mathbf{L} = \nabla\mathbf{v}$ is not objective.

Examining the rate of deformation tensor, on the other hand, one finds:

$$\begin{aligned} \mathbf{D}^* &= \frac{1}{2}(\mathbf{L}^* + (\mathbf{L}^*)^T) \\ &= \frac{1}{2}[\mathbf{QLQ}^T + \dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}(\mathbf{L})^T\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{Q}}^T] \end{aligned}. \tag{2.109}$$

One can also show that

$$\dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{Q}}^T = \frac{d}{dt}[\mathbf{QQ}^T] = \frac{d}{dt}[\mathbf{I}] = 0, \tag{2.110}$$

so substituting this result into (2.109) gives

$$\mathbf{D}^* = \frac{1}{2}\mathbf{Q}[\mathbf{L} + \mathbf{L}^T]\mathbf{Q}^T = \mathbf{QDQ}^T, \tag{2.111}$$

which shows us that $\mathbf{D}$ is objective.

Therefore, we have a spatial rate-of-strain object, $\mathbf{D}$, that is objective. The question arises about whether corresponding reference measures of rate are objective. It turns out that such material rates are automatically objective, since they do not change when superimposed rotations occur spatially. Consider, for example, the right Cauchy-Green tensor $\mathbf{C}$:

$$\mathbf{C}^* = (\mathbf{F}^*)^T(\mathbf{F}^*) = \mathbf{F}^T\mathbf{Q}^T\mathbf{QF} = \mathbf{C}. \tag{2.112}$$

In view of (2.112) it is obvious that

$$\dot{\mathbf{C}}^* = \dot{\mathbf{C}}. \tag{2.113}$$

Turning our attention to stress rates, let us examine the material time derivative of the Cauchy stress $\mathbf{T}$:

$$\dot{\mathbf{T}} = \left[\frac{d}{dt}(\mathbf{T}\circ\varphi_t)\right] \bullet \varphi_t^{-1} = \left(\frac{\partial\mathbf{T}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{T}\right). \tag{2.114}$$

Now $\mathbf{T}$ is itself objective by its very definition as a tensoral quantity. Thus we can write

$$\mathbf{T}^* = \mathbf{QTQ}^T. \tag{2.115}$$

Computing the material time derivative of (2.115) we find

$$\dot{\mathbf{T}}^* = \dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^T + \mathbf{Q}\mathbf{T}\dot{\mathbf{Q}}^T. \tag{2.116}$$

Since the first and third terms on the right-hand side of (2.116) do not, in general, cancel, we see that the material time derivative of the Cauchy stress $\mathbf{T}$ is not objective.

It, therefore, becomes critical, when a constitutive description requiring a stress rate is to be formulated, to consider a frame indifferent measure of stress rate. A multitude of such rates have been contrived; the interested reader is encouraged to consult [Marsden, J.E. and Hughes, T.J.R., 1983] for a highly theoretical treatment. For our discussion here we consider two such rates, especially prevalent in the literature: the Jaumann rate and the Green-Naghdi rate. Both rates rely on roughly the same physical idea: rather than taking the derivative of the Cauchy stress itself, we rotate the object from the spatial frame before taking the time derivative, so that the reference frame in which the time derivative is taken is the same for all frames related by the transformation (2.104).

For example, let us consider the Jaumann rate of stress, which we denote here as $\hat{\mathbf{T}}$. Its definition is given as follows:

$$\hat{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W}. \tag{2.117}$$

We can verify that this rate of stress is truly objective by direct calculation, by considering the object as it would appear to observer *:

$$\hat{\mathbf{T}}^* = \dot{\mathbf{T}}^* - \mathbf{W}^*\mathbf{T}^* + \mathbf{T}^*\mathbf{W}^*. \tag{2.118}$$

The quantity $\dot{\mathbf{T}}^*$ is given by (2.116), $\mathbf{T}^*$ is given by (2.115), and $\mathbf{W}^*$ can be computed with the aid of (2.108) and (2.111):

$$\mathbf{W}^* = \mathbf{L}^* - \mathbf{D}^* = \mathbf{Q}\mathbf{L}\mathbf{Q}^T + \dot{\mathbf{Q}}\mathbf{Q}^T - \mathbf{Q}\mathbf{D}\mathbf{Q}^T. \tag{2.119}$$

Substituting these quantities into (2.118) we find

$$\begin{aligned}\hat{\mathbf{T}}^* = {} & \dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^T + \mathbf{Q}\mathbf{T}\dot{\mathbf{Q}}^T \\ & -(\mathbf{Q}\mathbf{L}\mathbf{Q}^T + \dot{\mathbf{Q}}\mathbf{Q}^T - \mathbf{Q}\mathbf{D}\mathbf{Q}^T)\mathbf{Q}\mathbf{T}\mathbf{Q}^T \\ & + \mathbf{Q}\mathbf{T}\mathbf{Q}^T(\mathbf{Q}\mathbf{L}\mathbf{Q}^T + \dot{\mathbf{Q}}\mathbf{Q}^T - \mathbf{Q}\mathbf{D}\mathbf{Q}^T)\end{aligned}. \tag{2.120}$$

Canceling terms and using the fact that $\dot{\mathbf{Q}}\mathbf{Q}^T = -\mathbf{Q}\dot{\mathbf{Q}}^T$, we can simplify (2.120) to

$$\hat{\mathbf{T}}^* = \mathbf{Q}[\dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W}]\mathbf{Q}^T = \mathbf{Q}\hat{\mathbf{T}}\mathbf{Q}^T, \tag{2.121}$$

which ensures us that, indeed, $\hat{\mathbf{T}}$ is objective.

In consideration of the Green-Naghdi rate we, perhaps, gain more insight into how objective rates can be designed. The Green-Naghdi rate of Cauchy stress is defined via:

$$\tilde{\mathbf{T}} = \mathbf{R}\dot{\boldsymbol{T}}\mathbf{R}^{\mathsf{T}}, \tag{2.122}$$

where $\mathbf{R}$ is the rotation tensor from the polar decomposition of $\mathbf{F}$, and $\boldsymbol{T}$ is the rotated Cauchy stress defined in (2.57).

Let us examine how the rotation tensor $\mathbf{R}$ transforms. Recalling Eq. (2.106) we can write

$$\mathbf{F}^*=\mathbf{R}^*\mathbf{U}^* = \mathbf{QF} = \mathbf{QRU}. \tag{2.123}$$

We now note two things: first, that the product $\mathbf{QR}$ is itself a proper orthogonal tensor and second, that the polar decomposition is unique for a given deformation gradient. Therefore, comparing the second and fourth terms of (2.123), we must conclude:

$$\mathbf{U}^* = \mathbf{U}, \tag{2.124}$$

and

$$\mathbf{R}^* = \mathbf{QR}. \tag{2.125}$$

Using Eqs. (2.125) and (2.122) we can compute:

$$\tilde{\mathbf{T}}^* = \mathbf{R}^*\dot{\boldsymbol{T}}^*\mathbf{R}^{*\mathsf{T}} = \mathbf{QR}\dot{\boldsymbol{T}}^*\mathbf{R}^{\mathsf{T}}\mathbf{Q}^{\mathsf{T}}. \tag{2.126}$$

Returning to the definition of $\boldsymbol{T}$ in Eq. (2.57) and incorporating Eqs. (2.115) and (2.125), we can write

$$\boldsymbol{T}^*=\mathbf{R}^{*\mathsf{T}}\mathbf{T}^*\mathbf{R}^*=\mathbf{R}^{\mathsf{T}}\mathbf{Q}^{\mathsf{T}}(\mathbf{QTQ}^{\mathsf{T}})\mathbf{QR} = \mathbf{R}^{\mathsf{T}}\mathbf{TR} = \boldsymbol{T}. \tag{2.127}$$

Therefore, the rotated stress tensor appears exactly the same in both frames of reference. It follows that

$$\dot{\boldsymbol{T}}^* = \dot{\boldsymbol{T}}, \tag{2.128}$$

which, when substituted into (2.126), gives

$$\tilde{\mathbf{T}}^*=\mathbf{QR}\dot{\boldsymbol{T}}\mathbf{R}^{\mathsf{T}}\mathbf{Q}^{\mathsf{T}} = \mathbf{Q}\tilde{\mathbf{T}}\mathbf{Q}^{\mathsf{T}}, \tag{2.129}$$

which is recognized as nothing more than the properly objective transformation of $\tilde{\mathbf{T}}$.

One may note that result (2.128) gives considerable insight into how objective rates can be constructed. In the current case we transform the stress into the rotated configuration before computing its time derivative, and then transform the result back to the spatial configuration. Since the rotated stress is exactly the same for all reference frames related by (2.104), taking the time derivative of it and then transforming produces an objective object. This idea can be generalized as follows: construction of an objective rate of stress is achieved by considering the time derivative of a stress measure defined in a coordinate system that is rotating about some set of axes. In fact, one can show that the Jaumann stress rate can be similarly interpreted.

Finally, the Green-Naghdi rate can be manipulated further to a form resembling more closely the form given for the Jaumann rate (Eq. (2.117)). We may write

$$\begin{aligned} \tilde{\mathbf{T}} &= \mathbf{R}\frac{d}{dt}(\mathbf{R}^{T}\mathbf{T}\mathbf{R})\mathbf{R}^{T} \\ &= \mathbf{R}\dot{\mathbf{R}}^{T}\mathbf{T} + \dot{\mathbf{T}} + \mathbf{T}\dot{\mathbf{R}}\mathbf{R}^{T}, \\ &= \dot{\mathbf{T}} + \boldsymbol{L}^{T}\mathbf{T} + \mathbf{T}\boldsymbol{L} \\ &= \dot{\mathbf{T}} + \mathbf{T}\boldsymbol{L} - \boldsymbol{L}\mathbf{T} \end{aligned} \tag{2.130}$$

where we have used Eq. (2.42) to define $\boldsymbol{L}$, recalling also that this object is skew.

Two-Dimensional Formulations

Introduction

In this section we very briefly discuss the adaptation of the three-dimensional framework to two-dimensional problems. We consider three cases of primary interest: plane strain, plane stress, and axisymmetry.

Plane Strain

The so-called plane strain assumption is appropriate when the following conditions hold: 1) the object of interest can be geometrically described in a two-dimensional manner (for example, by considering a cross section of a very long object); 2) once so idealized, no loads on the structure act in the direction normal to the two-dimensional plane selected; 3) no significant displacement occurs normal to this plane; and 4) the variation of any quantity (stress, strain, displacement, etc.) in the direction normal to the plane can be neglected. Conditions 3) and 4) require that all out-of-plane strain components be zero, giving rise to the name *plane strain*.

The reader should refer to Figure 2.5 for the notational framework we will use. We associate the third index, $i = 3$, (i.e., the z-coordinate) with the out-of-plane direction. All of the continuum mechanical concepts we have developed for the three-dimensional case can then be straightforwardly applied to the current situation. We note that in two dimensions, one simply considers the large deformation boundary value problem summarized in Large Deformation Problems to be defined over a two-dimensional domain with the unknown motion φ having two components rather than three.

Note that, in general, it is necessary, however, to keep track of some stress components associated with the third dimension. This comes about due to the coupling between the in-plane strains and the out-of-plane stresses. For example, considering infinitesimal elasticity for a moment, we have the following strain components equal to zero:

$$E_{13} = E_{23} = E_{33} = 0. \tag{2.131}$$

If the elastic response is isotropic, we can use Eqs. (1.59) and (1.62) to conclude that

$$T_{13} = T_{23} = 0, \tag{2.132}$$

but also that

$$T_{33} = \lambda(E_{11} + E_{22}) \neq 0. \tag{2.133}$$

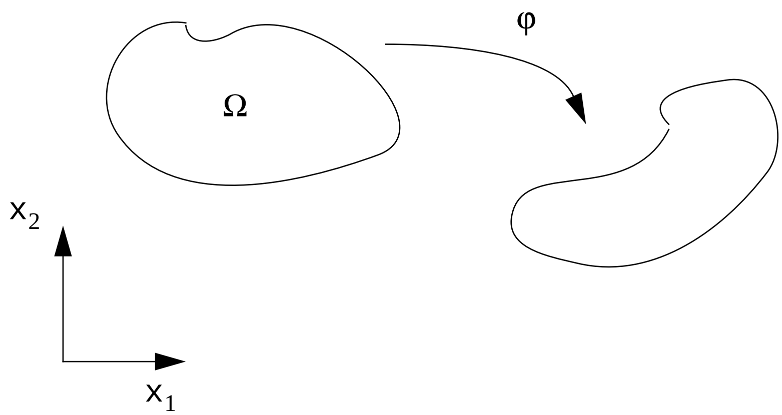

Figure 2.5 Two-dimensional notation for plane stress and plane strain cases.

Plane Stress

The plane stress assumption is appropriate when the following set of circumstances hold: 1) the object of interest can be geometrically described in a two-dimensional manner; 2) once so idealized, no loads on the structure act in the direction normal to the two-dimensional plane selected; 3) no significant internal stress is generated in the direction normal to this plane; and 4) the variation of stress, strain, and in-plane displacement in the direction normal to the plane can be neglected. Condition 3), in particular, makes this idealization most appropriate for thin, flat objects subject to in-plane loads. The fact that nonzero stresses are assumed to lie within the plane gives rise to the name *plane stress*.

The notation given in Figure 2.5 is appropriate for this class of problems, and as was the case in plane strain, we simply specify the problem as a two-dimensional boundary value problem solving for the two-vector φ. Again, however, in describing the constitutive relations some knowledge of the third dimension must be maintained. Considering again the linear elastic case for simplicity, we have

$$T_{13} = T_{23} = T_{33} = 0, \tag{2.134}$$

from which we can conclude (for isotropy) that

$$E_{13} = E_{23} = 0, \tag{2.135}$$

but also that

$$E_{33} = \frac{-\lambda(E_{11} + E_{22})}{\lambda + 2\mu} \neq 0, \tag{2.136}$$

in general. Particularly when formulating plasticity problems, the out-of-plane straining is important to include as we shall see in later work.

Axisymmetry

An axisymmetric formulation is useful when an object possesses an axis of symmetry, and when the loading, boundary conditions, and response are invariant with respect to rotation about this axis. Under these circumstances it is convenient to construct a coordinate system, (r, z), as shown in Figure 2.6.

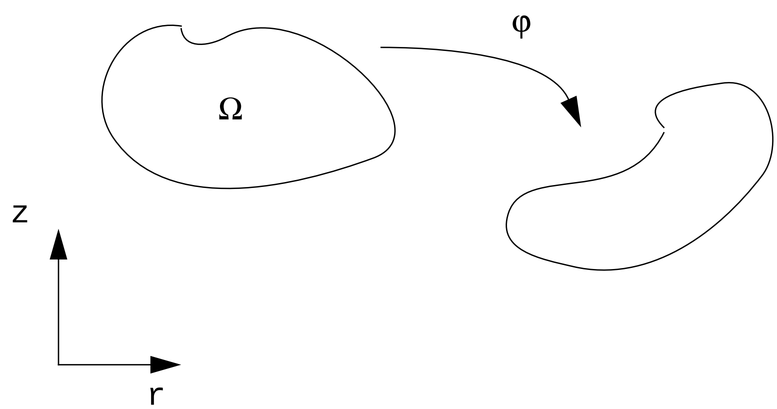

Figure 2.6 Notation for an axisymmetric problem (z is the axis of symmetry). The actual three-dimensional object is obtained by rotating the above cross sections 360 degrees about the z-axis.

An in-depth treatment of axisymmetry is beyond the scope of our current treatment. The main idea is that our coordinate system is no longer Cartesian but is instead curvilinear. For reference we consider again the infinitesimal case. We consider a displacement vector,

$$\mathbf{u} = \begin{bmatrix} u_r \\ u_z \end{bmatrix}, \tag{2.137}$$

and find that the appropriate expressions of strain are now

$$\begin{aligned} E_{rr} &= \frac{\partial u_r}{\partial r}, \quad E_{r\theta} = 0, \quad E_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \\ E_{\theta\theta} &= \frac{u_r}{r}, \quad E_{\theta z} = 0, \quad E_{zz} = \frac{\partial u_z}{\partial z} \end{aligned} . \tag{2.138}$$

The stress-strain relations are still as given by Eq. (1.59), i.e.

$$\mathbf{T} = \mathbf{C}:\mathbf{E}. \tag{2.139}$$

The differential equations of equilibrium do need to be rewritten, however, due to the special form of the stress divergence resulting from the curvilinear coordinate system. One finds the following to be the appropriate expressions of linear momentum balance for axisymmetric problems:

$$\begin{aligned} \frac{\partial T_{rr}}{\partial r} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr} + T_{\theta\theta}}{r} + f_r &= \rho \ddot{u}_r \\ \frac{\partial T_{rz}}{\partial r} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{rz}}{r} + f_z &= \rho \ddot{u}_z \end{aligned} . \tag{2.140}$$

Structural Components

Introduction

Most discussion in this report has been primarily concerned with the description of deformation and stress in fully three-dimensional bodies. It is frequently desirable in solid mechanics to describe entities that are comparatively thin in at least one spatial direction and perhaps in two. The former case is commonly referred to as a shell or plate (depending on whether the entity is initially curved or flat), and the second case is referred to as a beam or truss (depending upon whether bending is to be considered). In this section we briefly discuss how the continuum mechanical framework we have constructed can be adapted to these situations.

The Degenerated Solid Approach

We consider initially a thin plate- or shell-like object, described schematically as shown in Figure 2.7. We consider that there is one spatial dimension, the through-the-thickness direction, that is much smaller than the characteristic in-plane dimensions of the object.

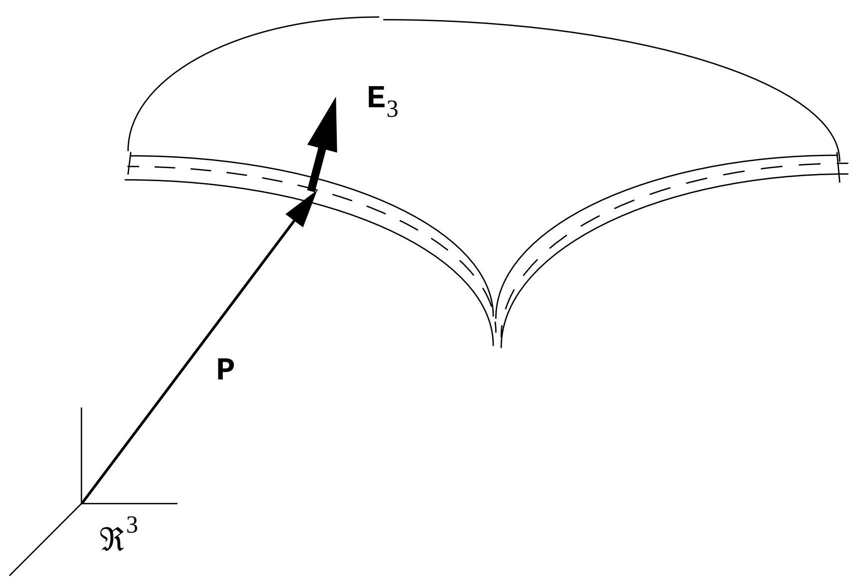

Figure 2.7 Schematic of a generic plate or shell object shown in the reference configuration.

One could consider the ordinary three-dimensional formalism to apply pointwise within this solid, leading to a boundary value problem written in terms of all three displacement components u_i. When analyzing shells, however, we become interested in writing the equations in terms of only the midsurface position, denoted as $\mathbf{P}$ in the figure, and the rotations of unit vectors $\mathbf{E}_3$ that are normal to this surface in the reference configuration. We can, therefore, express any reference point $\mathbf{X}$ in the shell in terms of the midsurface position $\mathbf{P}$ and the unit vector:

$$\mathbf{X} = \mathbf{P} + Z\mathbf{E}_3, \tag{2.141}$$

where Z is a through-the-thickness coordinate ranging between $-\frac{t}{2}$ and $\frac{t}{2}$. The quantity t is the local thickness of the shell referenced by $\mathbf{P}$.

As readers familiar with solid mechanics will be aware, the equations governing structural objects are conveniently written in terms of so-called stress resultants, or net moments, torques, and forces, acting across cross sections. In the degenerated solid approach, one takes the fully three-dimensional equations of motion and performs through-the-thickness integration in terms of the appropriate coordinate (in this case Z) to obtain governing equations in terms of the midsurface displacements of points $\mathbf{P}$ and rotations of vectors $\mathbf{E}_3$.

If the deformation is infinitesimal, so that reference and current coordinates are the same and changes in the thickness t are insignificant, we obtain the shell equations of equilibrium by calculating

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} \left(\nabla \cdot \mathbf{T} + \mathbf{f} + -\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \right) dZ = 0. \tag{2.142}$$

The result will be a boundary value problem written in terms of midsurface displacements, rotations, and stress resultants. We will return to this approach in more detail when discussing finite element procedures for treating the shell and plate equations in a companion report.

Plates and Shells

In addition to the concept of a degenerated solid, another important aspect of plate and shell formulations is the specific kinematic description used to quantify displacement. Referring again to Figure 2.7, we describe the configuration mapping for any point in the shell in terms of the midsurface displacement Φ and the normal vector rotation $\mathbf{q}$:

$$\varphi(\mathbf{X}) = \Phi(\mathbf{P}) + Z\mathbf{q} \times \mathbf{E}_3 \tag{2.143}$$

where we have actually made two kinematic assumptions: first, that the through-the-thickness deformation is negligible so that Z is the same coordinate as in (2.141); and second, that normals to the midsurface (i.e., $\mathbf{E}_3$) remain straight, although not necessarily normal. This assumption leads to Mindlin plate theory where the rotation of vectors $\mathbf{E}_3$ with respect to the reference surface gives rise to transverse shear strains and stresses in the material.

Examining Eq. (2.143) we see that there are three dependent variables associated with the mapping Φ and three, in general, associated with the rotation $\mathbf{q}$. However, we generally discard the component of $\mathbf{q}$ producing rotation about $\mathbf{E}_3$. Thus, in general, there are five dependent variables we seek to find in a plate or shell boundary value problem.

Beams

Beams can be considered within this framework also by considering two transverse dimensions to be small when compared to the remaining one (i.e., beam length). Thus, rather than degenerating in one spatial dimension to obtain the resultant-based equilibrium equations (as in (2.142)), we integrate in two. The result is usually a system with six dependent variables, described in terms of a one-dimensional object (a reference line, rather than surface, in this case). Three of these correspond to the x, y, and z-components of the reference line displacement, and the other three correspond to rotations.

Rigid Bodies

Finally, it will be desirable in some problems to be able to incorporate the equations of motion into a system in which some bodies or entities are rigid (i.e., no deformation is allowed to occur within some reference domain Ω). In this case, one has the followingset of governing equations from rigid body dynamics:

$$\begin{aligned}
\sum F_x &= Ma_x \\
\sum F_y &= Ma_y \\
\sum F_z &= Ma_z \\
\sum M_1 &= I_1\dot{\omega}_1-(I_2-I_3)\omega_2\omega_3 \\
\sum M_2 &= I_2\dot{\omega}_2-(I_3-I_1)\omega_3\omega_1 \\
\sum M_3 &= I_3\dot{\omega}_3-(I_1-I_2)\omega_1\omega_2
\end{aligned}, \qquad (2.144)$$

where the subscripts x, y, and z refer to global (reference) coordinates, and subscripts 1, 2, and 3 refer to the principal directions of the inertia tensor fixed to the solid. The quantities I_1, I_2, and I_3 are the principal values of the inertia tensor, and the ω's are the components of angular velocity in the principle directions. We will return to the implementation of these equations when discussing finite element methods in a companion report.

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